Stat 306:
Finding Relationships in Data.
Lecture 4
2.2 (continued) + 2.5 Intervals for simple linear regression

t-test

Age vs. Money

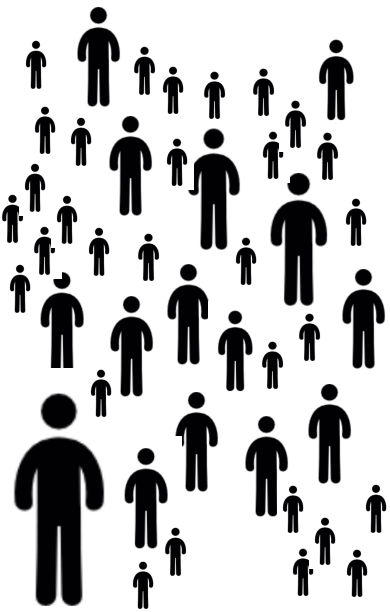
Dependent variable

χ → old (0)
χ → young (1)

Independent variable

γ → dollars ($)
In bank account

Population



Population parameters

μ_0 , μ_1 , σ^2

Hypothesis Test

$H_0 : \mu_0 = \mu_1$

$H_1 : \mu_0 \neq \mu_1$

Sample statistics

$\bar{y}_0 = 56$
$\bar{y}_1 = 27$
$\bar{y}_0 - \bar{y}_1 = 29$
$s_p = 10.81$

t = 2.68, *df* = 7
p-value = 0.03
95% C.I. = [3.4, 54.6]

Sample, n=9

χ	γ
old	71
old	54
old	43
young	45
young	21
young	11
young	30
young	45
young	10

linear regression

Age vs. Money

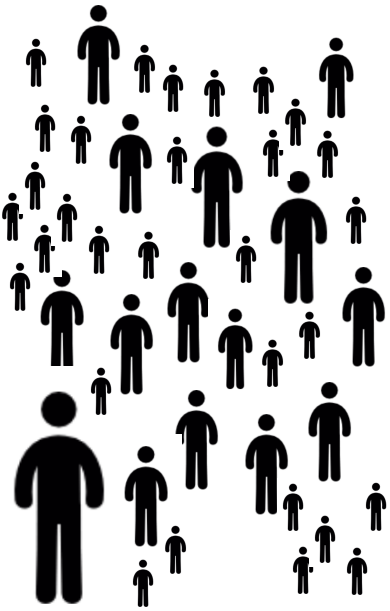
PREDICTOR *variable*

$X \longrightarrow$ Age in Years

RESPONSE *variable*

$Y \longrightarrow$ dollars ($) In bank account

Population



Population parameters

$\beta_0, \beta_1, \sigma^2$

Hypothesis Test

$H_0 : \beta_1 = 0$

$H_1 : \beta_1 \neq 0$

Sample statistics

$b_0 = 17.7$
$b_1 = 0.55$
$s = 15.5$
$R^2 = 0.49$

For parameter β_1 :

95% C.I. = [0.05, 1.05]
p-value = 0.036

Sample, n=9

x	y
82	71
45	54
71	43
22	45
29	21
9	11
12	30
18	45
24	10

linear regression

Age vs. Money

fixed value

PREDICTOR *variable*

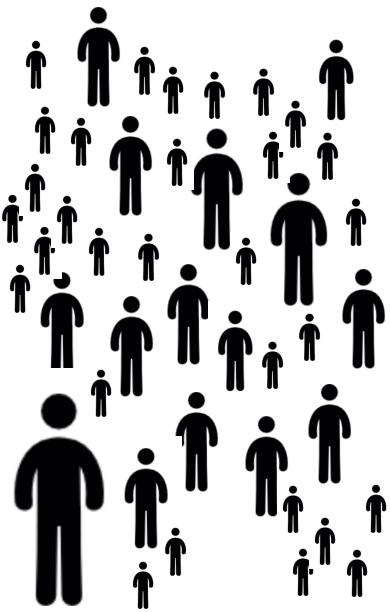
X ⟶ Age in Years

Random variable

RESPONSE *variable*

Y ⟶ dollars ($) In bank account

Population



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Section 2.2 - Statistical linear regression model

What is a random variable?

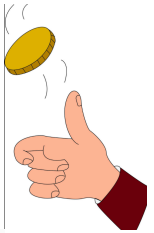
"A random variable, Y, is a variable whose possible values are numerical outcomes of a random phenomenon."

For a Random Variable, Y, we typically want to talk about the Expectation and Variance.

Example 1: $E[Y] = 3.5$ $\quad$ $Var(Y) = 2.92$

Example 2: $E[Y] = \theta$ $\quad$ $Var[Y] = \theta\,(1-\theta)$

Example 3: $E[Y] = \beta_0 + \beta_1 X$ $\quad$ $Var(Y) = \sigma^2$



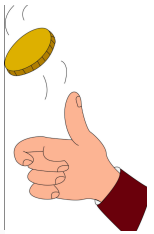
Questions?

Section 2.2 - Statistical linear regression model

What is a random variable?

"A random variable, Y, is a variable whose possible values are numerical outcomes of a random phenomenon."

We can ONLY talk about the Expectation and Variance of Y, if Y is a random variable.



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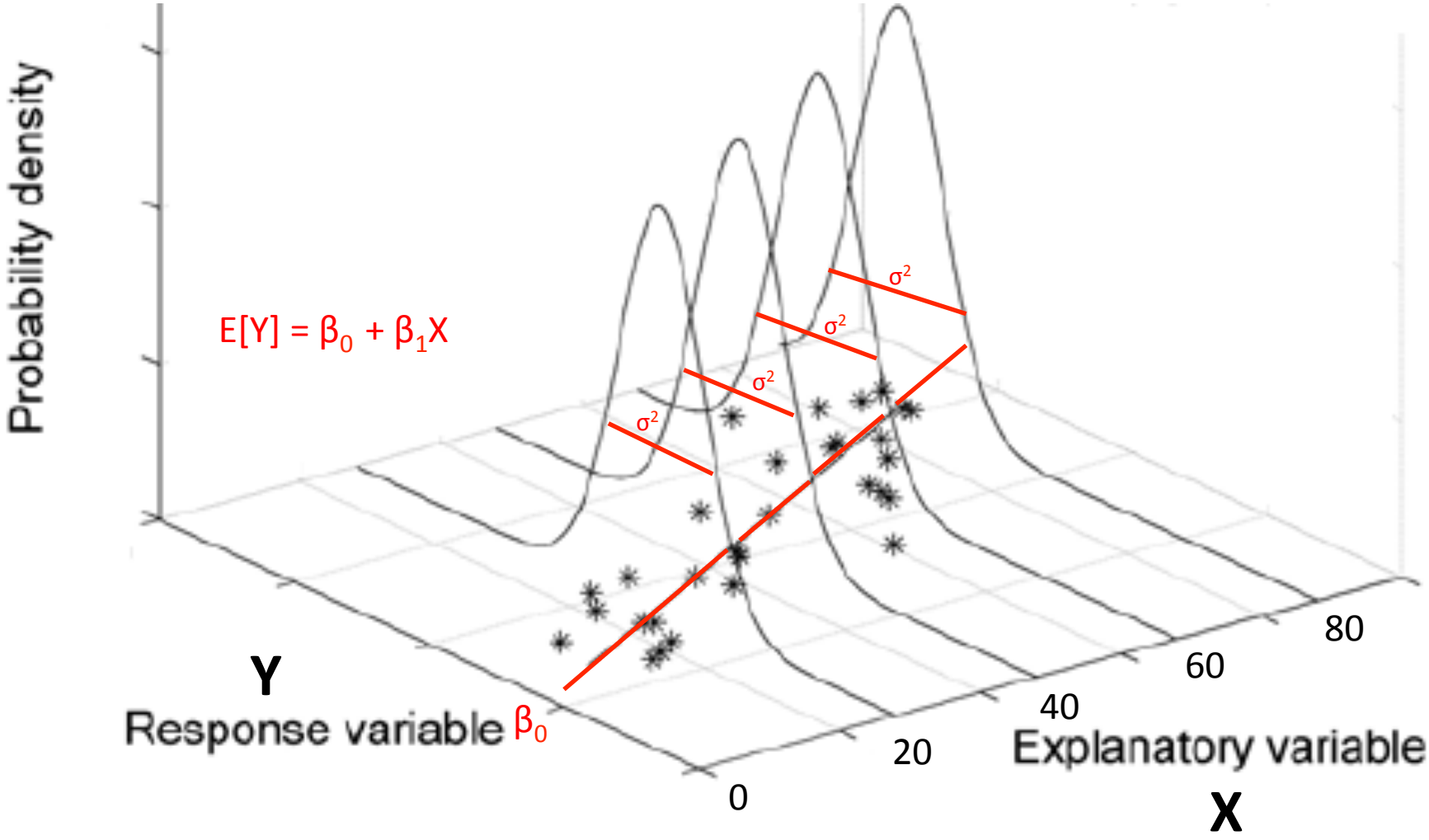
Section 2.2 - Statistical linear regression model

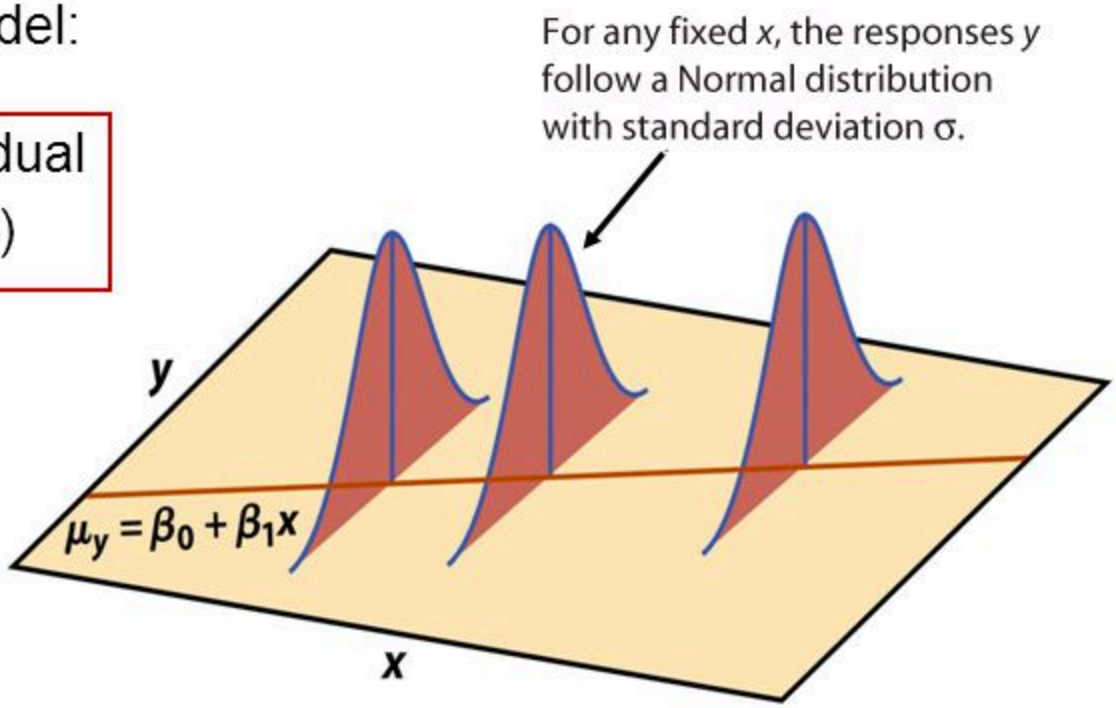


Simple linear regression model

In the population, the linear regression equation is $E(y) = \beta_0 + \beta_1 x$.

Sample data then fits the model:

Data =	fit	+	residual
y_i =	$(\beta_0 + \beta_1 x_i)$	+	(ε_i)



where the ε_i are **independent** and **Normally** distributed $N(0, \sigma)$.

Linear regression assumes **equal standard deviation of *y*** (σ is the same for all values of x).

Credit: Julien Stamp
http://slideplayer.com/slide/2342871/

Section 2.2 - Statistical linear regression model

Background on interval estimation

An interval estimate is an interval of plausible values for unknown population quantity, computed based on the observed data. A longer interval has a large confidence level.

- θ is generic (scalar) parameter to be estimated; for example, population mean μ, proportion π, regression coefficient β.
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where $se(\hat{\theta})$ is an estimate of $\sqrt{\mathrm{Var}(\hat{\Theta})}$, and c depends on the confidence level. Typically $se(\hat{\theta}) = O(1/\sqrt{n})$ so that interval gets smaller as sample size n increases.

- Examples of 95% confidence intervals: with $\overline{y} = \hat{\mu}$,

$$\overline{y} \pm t_{n-1,0.975} \times se(\overline{y}); \quad se(\overline{y}) = s_y/\sqrt{n} \tag{2.34}$$

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$$\hat{\mu} = \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$

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$$\hat{\beta}_1 = b_1 = r_{xy} \frac{s_y}{s_x}$$

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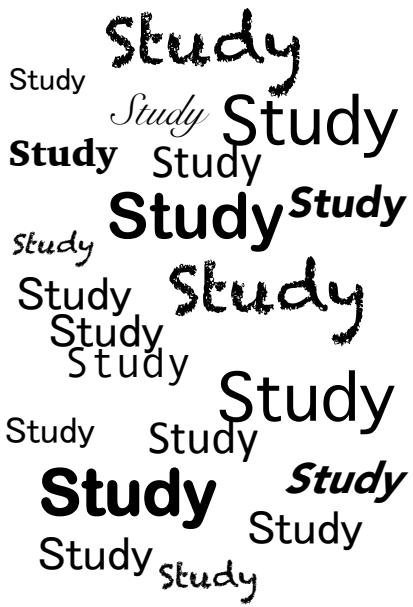
RANDOM variable

$\hat{\Theta}$ ⟶ Estimate from a study

Population



Sample

Study $\hat{\theta}_1$

Study $\hat{\theta}_2$

Study $\hat{\theta}_3$

Study $\hat{\theta}_4$

Study $\hat{\theta}_5$

Study $\hat{\theta}_6$

Hence we can consider:

$$\mathrm{E}[\hat{\Theta}]$$

$$\mathrm{Var}[\hat{\Theta}]$$

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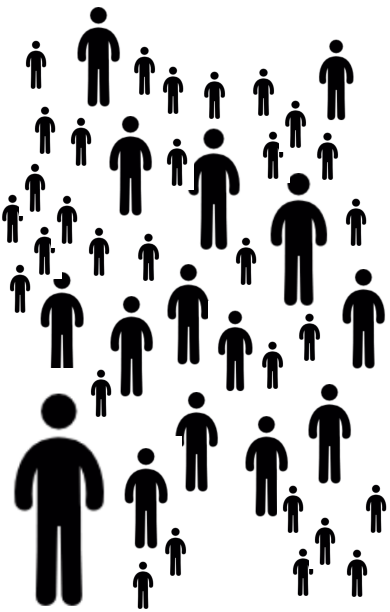

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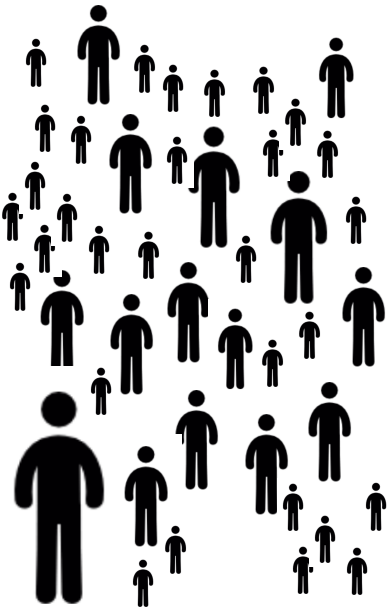

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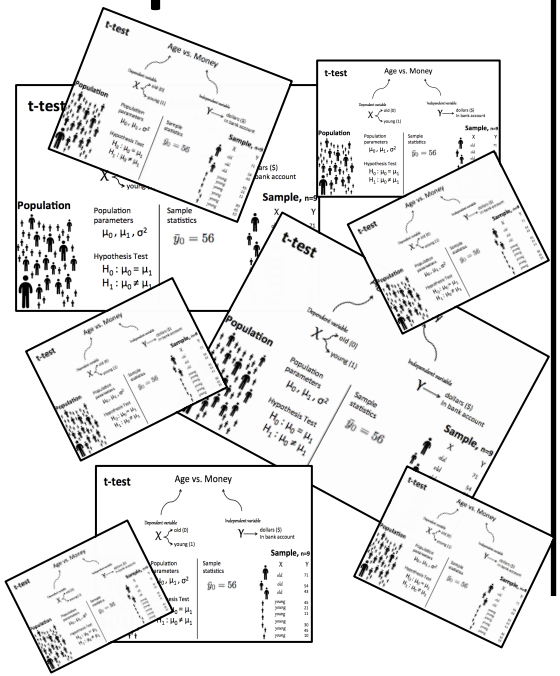

Sample

$$\bar{y}_0 = 46$$
$$\bar{y}_0 = 86$$
$$\bar{y}_0 = 36$$
$$\bar{y}_0 = 66$$
$$\bar{y}_0 = 55$$

Hence we can consider:

$$\mathrm{E}[\bar{Y}_0] = \mu_0$$

$$\mathrm{Var}[\bar{Y}_0] = \sigma^2/n$$

http://www.zoology.ubc.ca/~whitlock/Kingfisher/SamplingNormal.htm

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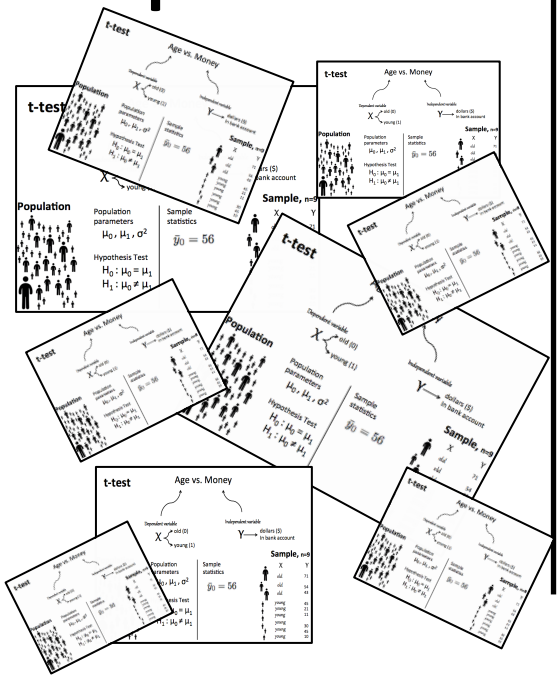

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The "standard error" is the error of the sample statistic with respect to estimating the population parameter. If **n** is very large, then **se** is very small. As n increases linearly, **se** decreases by 1/sqrt(n).

https://en.wikipedia.org/wiki/Standard_error

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Typically the "confidence level" is 95% or 80% or 90%.

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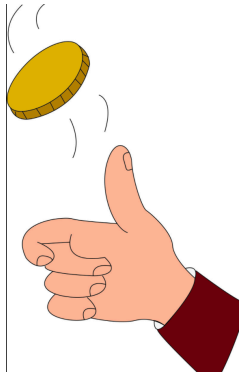
Section 2.2 - Statistical linear regression model

Example 2: All we can know about Y is that:

Pr(Y=0) = 1-θ

Pr(Y=1) = θ



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Rcode:

http://www.r-tutor.com/elementary-statistics/interval-estimation/interval-estimate-population-proportion

Population

Population parameters

θ

Hypothesis Test

$H_0 : \theta = 0.5$

$H_1 : \theta \neq 0.5$

Is the coin fair?

Sample statistics

$\hat{\theta} = 60/90 = 0.667$

We can also think of $\hat{\Theta}$ as a Random variable…. so:

$$\mathrm{Var}[\hat{\Theta}] = \theta(1-\theta)/n$$

Sample, n=90

	y
heads	0
tails	1
heads	0
tails	1
tails	1
heads	0
tails	1
…	
tails	1
tails	1

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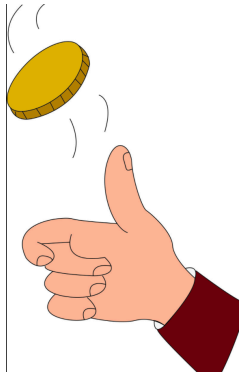

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	y
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Pr(Y=0) = 1-θ

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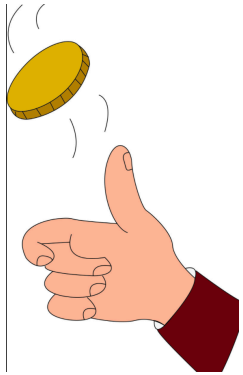


From the notes:

$\hat{\pi} \pm z_{0.975} \times se(\hat{\pi})$; $se(\hat{\pi}) = \sqrt{\hat{\pi}(1-\hat{\pi})}/\sqrt{n}$, $\mathrm{Var}(\hat{\Pi}) = \pi(1-\pi)/n$.

Rcode:

http://www.r-tutor.com/elementary-statistics/interval-estimation/interval-estimate-population-proportion

Population

Population parameters

θ

Hypothesis Test

$H_0 : \theta = 0.5$

$H_1 : \theta \neq 0.5$

Is the coin fair?

Sample, n=90

Sample statistics

$$\hat{\theta} = 60/90 = 0.667$$

$$se(\hat{\theta}) = \sqrt{\hat{\theta}(1-\hat{\theta})}/\sqrt{n}$$

= 0.05

Using the standard error, we can construct a 95% confidence interval:

$$[\hat{\theta} - z_{0.975}se(\hat{\theta}), \hat{\theta} + z_{0.975}se(\hat{\theta})]$$

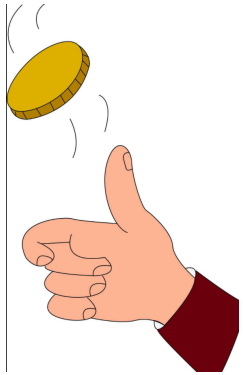
	y
heads	0
tails	1
heads	0
tails	1
tails	1
heads	0
tails	1
…	
tails	1
tails	1

Section 2.2 - Statistical linear regression model

Example 2: All we can know about Y is that:

Pr(Y=0) = 1-θ

Pr(Y=1) = θ



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Sample, n=90

Sample statistics

$\hat{\theta} = 60/90 = 0.667$

$se(\hat{\theta}) = \sqrt{\hat{\theta}(1-\hat{\theta})}/\sqrt{n}$

= 0.05

For 95% C.I. :

$[\hat{\theta} - z_{0.975} se(\hat{\theta}), \hat{\theta} + z_{0.975} se(\hat{\theta})]$

= [0.569, 0.764]

	y
heads	0
tails	1
heads	0
tails	1
tails	1
heads	0
tails	1
…	
tails	1
tails	1

- Questions?

Background on interval estimation

An interval estimate is an interval of plausible values for unknown population quantity, computed based on the observed data. A longer interval has a large confidence level.

- θ is generic (scalar) parameter to be estimated; for example, population mean μ, proportion π, regression coefficient β.
- $\hat{\theta}$ is an estimator of θ based on the data; for example, $\hat{\mu} = \overline{y}$.
- $\hat{\theta}$ is a realization of a random variable $\hat{\Theta}$ assuming a probability model. Hence $\mathrm{E}(\hat{\Theta})$ and $\mathrm{Var}(\hat{\Theta})$ can be considered.
- A confidence interval for a parameter θ commonly has the form

$$\hat{\theta} \pm c \times se(\hat{\theta}),$$

  where $se(\hat{\theta})$ is an estimate of $\sqrt{\mathrm{Var}(\hat{\Theta})}$, and c depends on the confidence level. Typically $se(\hat{\theta}) = O(1/\sqrt{n})$ so that interval gets smaller as sample size n increases.
- Examples of 95% confidence intervals: with $\overline{y} = \hat{\mu}$,

$$\overline{y} \pm t_{n-1,0.975} \times se(\overline{y}); \quad se(\overline{y}) = s_y/\sqrt{n} \tag{2.34}$$

$$\hat{\pi} \pm z_{0.975} \times se(\hat{\pi}); \quad se(\hat{\pi}) = \sqrt{\hat{\pi}(1-\hat{\pi})}/\sqrt{n}. \tag{2.35}$$

  Here $z_{0.975}$ is the upper 0.975 quantile of the standard normal distribution and $t_{\nu,0.975}$ is the upper 0.975 quantile of the Student t distribution with degree of freedom parameter ν. (2.35) is based on an approximation that is valid for large n.

linear regression

Age vs. Money

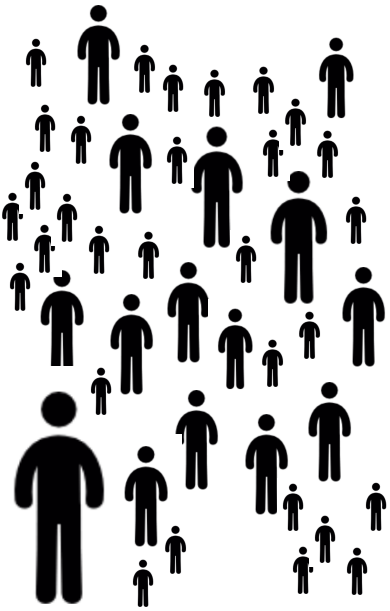
PREDICTOR *variable*

χ ⟶ Age in Years

RESPONSE *variable*

γ ⟶ dollars ($) In bank account

Population



Population parameters

β_0 , β_1 , σ^2

Hypothesis Test

$H_0 : \beta_1 = 0$

$H_1 : \beta_1 \neq 0$

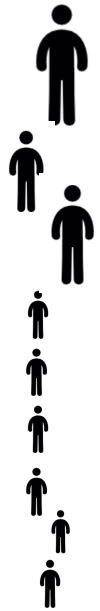
Sample statistics

b_0 = 17.7
b_1 = 0.55
s = 15.5
R^2 = 0.49

For parameter β_1 :

95% C.I. = [0.05, 1.05]
p-value = 0.036

Sample, n=9



χ	γ
82	71
45	54
71	43
22	45
29	21
9	11
12	30
18	45
24	10

linear regression

Age vs. Money

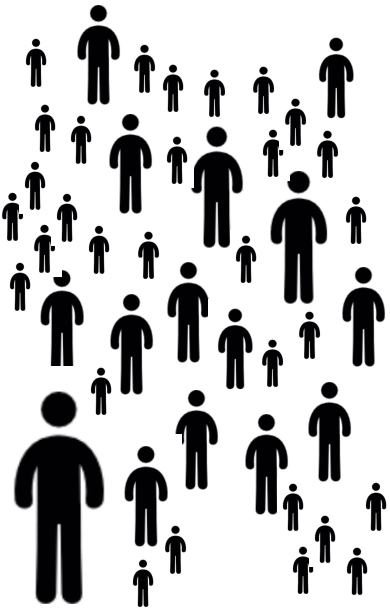
PREDICTOR *variable*

X ⟶ Age in Years

RESPONSE *variable*

Y ⟶ dollars ($) In bank account

Population



Population parameters

$\beta_0, \beta_1, \sigma^2$

Hypothesis Test

$H_0 : \beta_1 = 0$

$H_1 : \beta_1 \neq 0$

Sample statistics

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linear regression

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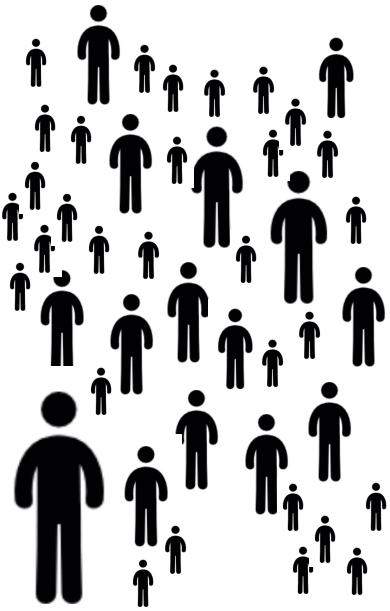
3. Define $se(b_1)$ as an estimate of $sqrt(Var(B_1))$
4. 95% C.I. = $[\, b_1 - c^*se(b_1) \,,\, b_1 + c^*se(b_1) \,]$

Population



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2.5 Intervals for simple linear regression

The following is an inference result from introductory statistics (but the proof is given in a mathematical statistics course). If the values $y_1, \ldots, y_n$ are realizations from a random sample from a normal population with mean μ and variance σ^2, then the 95% confidence interval for μ is:

(2.39) $$\overline{y} \pm t_{n-1,0.975} \times se(\overline{y}); \quad se(\overline{y}) = s/\sqrt{n},$$

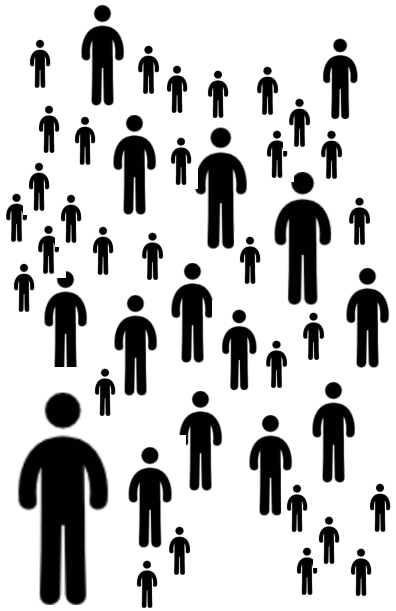
where $t_{\nu,0.975}$ is the 0.975 quantile of the Student t distribution with degree of freedom parameter ν. This means area to the left of $t_{\nu,0.975}$ under the t_ν density curve is 0.975. *Some statistics books denote this critical value as $t_{0.025,\nu}$, to indicate the area to the right is 0.025.*

“from introductory statistics…”

RANDOM *variable*

$$Y \sim Normal(\mu, \sigma^2)$$

Population



Population parameters

μ , σ^2

Hypothesis Test

$H_0 : \mu = 0$

$H_1 : \mu \neq 0$

Sample statistics

$$\bar{y} = n^{-1} \sum_{i=1}^{n} y_i$$

$$s = \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2 / (n-1)}.$$

We can also think of $\bar{Y}$ as a Random variable…. so:

$$\mathrm{Var}[\bar{Y}] = \sigma^2 / n$$

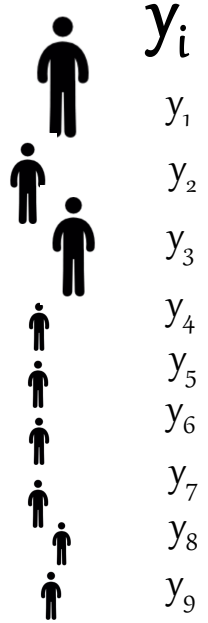
$se(\hat{\theta})$ is an estimate of $\sqrt{\mathrm{Var}(\hat{\Theta})}$

$$se(\bar{y}) = s/\sqrt{n},$$

the 95% confidence interval for μ is:

$$\bar{y} \pm t_{n-1,0.975} \times se(\bar{y});$$

Sample, n=9



"from introductory statistics…"

RANDOM *variable*

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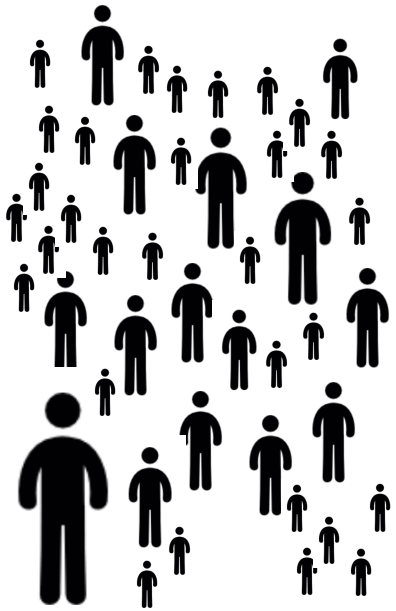
Population



Population parameters

μ , σ^2

Hypothesis Test

$H_0 : \mu = 0$

$H_1 : \mu \neq 0$

Sample statistics

$$\bar{y} = n^{-1} \sum_{i=1}^{n} y_i$$

$$s = \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2 / (n-1)}.$$

Step1

We can also think of $\bar{Y}$ as a Random variable…. so:

Step2

$$\mathrm{Var}[\bar{Y}] = \sigma^2 / n$$

Step3

$se(\hat{\theta})$ is an estimate of $\sqrt{\mathrm{Var}(\hat{\Theta})}$

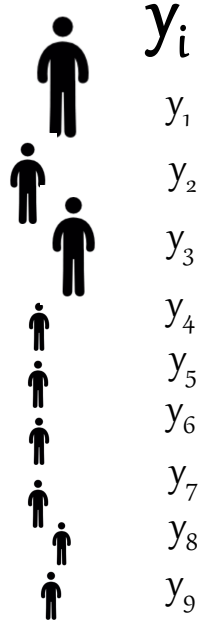
$$se(\bar{y}) = s/\sqrt{n},$$

Step4

the 95% confidence interval for μ is:

$$\bar{y} \pm t_{n-1,0.975} \times se(\bar{y});$$

Sample, n=9



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Rcode

2.5.2 Derivations

linear regression

Steps to get 95% C.I. for b_1

1. Consider the sample statistic
   b_1 as the random variable B_1

2. Determine Var[B_1]

3. Define se(b_1) as an estimate of sqrt(Var(B_1))

4. 95% C.I. for parameter β_1 = [b_1 - c*se(b_1) , b_1 + c*se(b_1)]

2.5.2 Derivations

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2.5.2 Derivations

The standard errors come from the variances when the estimators are considered as random variables. $\hat{\beta}_1$ as a random variable $\hat{B}_1$ is:

(2.50)
$$\hat{B}_1 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})Y_i}{(n-1)s_x^2} = \sum_{i=1}^{n} a_i Y_i,$$

where

(2.51)
$$a_i = (x_i - \overline{x})/[(n-1)s_x^2].$$

2.5.2 Derivations

$$b_1 = r_{xy}\frac{s_y}{s_x}$$

Recall: $r_{xy} = \frac{s_{xy}}{s_x s_y}$

$$= \frac{s_{xy}}{s_x s_y}\frac{s_y}{s_x}$$

Recall: $s_{xy} = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}).$

$$= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x^2}$$

$$= \frac{\sum_{i=1}^{n}(y_i)(x_i - \bar{x})}{(n-1)s_x^2} - \frac{\sum_{i=1}^{n}(\bar{y})(x_i - \bar{x})}{(n-1)s_x^2}$$

$$= \frac{\sum_{i=1}^{n}(y_i)(x_i - \bar{x})}{(n-1)s_x^2} - \bar{y}\frac{\sum_{i=1}^{n}(x_i - \bar{x})}{(n-1)s_x^2}$$

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$$= \frac{\sum_{i=1}^{n}(y_i)(x_i-\bar{x})}{(n-1)s_x^2} - \frac{\sum_{i=1}^{n}(\bar{y})(x_i-\bar{x})}{(n-1)s_x^2}$$

$$= \frac{\sum_{i=1}^{n}(y_i)(x_i-\bar{x})}{(n-1)s_x^2} - \bar{y}\frac{\sum_{i=1}^{n}(x_i-\bar{x})}{(n-1)s_x^2}$$

0

2.5.2 Derivations

$$b_1 = r_{xy}\frac{s_y}{s_x}$$

$$= \frac{\sum_{i=1}^{n}(y_i)(x_i - \bar{x})}{(n-1)s_x^2}$$

$$= \sum_{i=1}^{n}\frac{(x_i - \bar{x})}{(n-1)s_x^2}(y_i)$$

2.5.2 Derivations

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$$= \frac{\sum_{i=1}^{n}(y_i)(x_i-\bar{x})}{(n-1)s_x^2}$$

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Step 1. Consider the sample statistic b_1 as the random variable B_1 :

$$B_1 = \sum_{i=1}^{n}\frac{(x_i-\bar{x})}{(n-1)s_x^2}(Y_i)$$

$$= \sum_{i=1}^{n} a_i Y_i \quad \text{, where:} \quad a_i = \frac{(x_i-\bar{x})}{(n-1)s_x^2}$$

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Step 2. Determine Var[B_1]

First, recall that for random variable Y_i, we have:

$$(2.33) \qquad Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2).$$

2.5.2 Derivations

Step 2. Determine Var[B_1]

First, recall that for random variable Y_i, we have:

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2.5.2 Derivations

Step 2. Determine Var[B_1]

First, recall that for random variable Y_i, we have:

$$Y_i \sim N(\beta_0 + \beta_1 x_i,\, \sigma^2). \tag{2.33}$$

Then from page 28 of the course notes, we have solutions for Var[B_1] and E[B_1] :

$\hat{B}_1 = \sum_{i=1}^{n} a_i Y_i$ as a random variable using $Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ with a_i as in (2.51).

$$\mathrm{E}(\hat{B}_1) = \sum_{i=1}^{n} a_i \mathrm{E}(Y_i) = \sum_{i=1}^{n} a_i(\beta_0 + \beta_1 x_i) = \beta_0 \sum_{i=1}^{n} a_i + \beta_1 \sum_{i=1}^{n} a_i x_i \tag{2.56}$$

$$= 0 + \beta_1 \sum_{i=1}^{n} \frac{(x_i - \overline{x})x_i}{[(n-1)s_x^2]} = \beta_1, \tag{2.57}$$

$$\mathrm{Var}(\hat{B}_1) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(Y_i) = \sigma^2 \sum_{i=1}^{n} a_i^2 = \sigma^2 \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{[(n-1)s_x^2]^2} \tag{2.58}$$

$$= \frac{\sigma^2}{(n-1)s_x^2}. \tag{2.59}$$

2.5.2 Derivations

Steps to get 95% C.I. for b_1

1. Consider the sample statistic
   b_1 as the random variable B_1

2. **Determine Var[B_1]** $= \frac{\sigma^2}{(n-1)s_x^2}.$

3. Define se(b_1) as an estimate of sqrt(Var(B_1))

4. 95% C.I. = [b_1 - c*se(b_1) , b_1 + c*se(b_1)]

2.5.2 Derivations

Steps to get 95% C.I. for b_1

1. Consider the sample statistic b_1 as the random variable B_1

2. Determine $\text{Var}[B_1] = \frac{\sigma^2}{(n-1)s_x^2}$.

3. **Define $se(b_1)$ as an estimate of $\text{sqrt}(\text{Var}(B_1))$**

$$se(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{n-1}\, s_x}$$

where:

$$\hat{\sigma} = \text{residual SD} = \left\{(n-2)^{-1}\sum_{i=1}^{n} e_i^2\right\}^{1/2}$$

2.5.2 Derivations

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we take c = $t_{n-2,0.975}$

2.5.2 Derivations

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we take c = $t_{n-2,0.975}$

Then we have :

95% C.I. for β_1 : $[b_1 - t_{n-2,0.975}\frac{\hat{\sigma}}{\sqrt{n-1}s_x}, \quad b_1 + t_{n-2,0.975}\frac{\hat{\sigma}}{\sqrt{n-1}s_x}]$

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Steps to get 95% C.I. for b_1

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we take c = $t_{n-2,0.975}$

Then we have :

95% C.I. for β_1 : $[b_1 - t_{n-2,0.975}\frac{\hat{\sigma}}{\sqrt{n-1}s_x}, \quad b_1 + t_{n-2,0.975}\frac{\hat{\sigma}}{\sqrt{n-1}s_x}]$

where: $\hat{\sigma} = \text{residual SD} = \left\{(n-2)^{-1}\sum_{i=1}^{n} e_i^2\right\}^{1/2}$ (also known as "s")

linear regression

Age vs. Money

fixed value

PREDICTOR *variable*

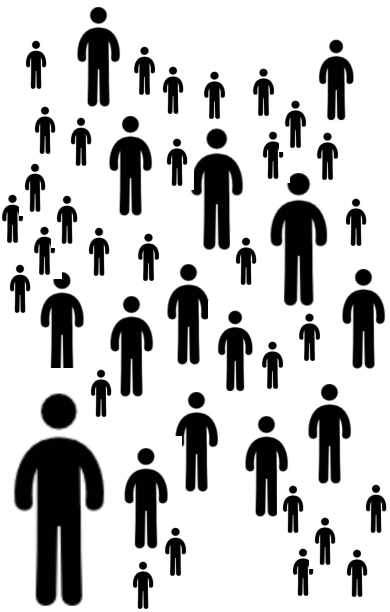
X ⟶ Age in Years

Random variable

RESPONSE *variable*

Y ⟶ dollars ($) In bank account

Population



Population parameters

$\beta_0, \beta_1, \sigma^2$

Hypothesis Test

$H_0 : \beta_1 = 0$

$H_1 : \beta_1 \neq 0$

Sample statistics

$b_0 = 17.7$

$b_1 = 0.55$

$s = 15.5$

$R^2 = 0.49$

For parameter β_1 :

95% C.I. = [0.05, 1.05]

p-value = 0.036

Sample, n=9

x	y
82	71
45	54
71	43
22	45
29	21
9	11
12	30
18	45
24	10

linear regression

Age vs. Money

fixed value

PREDICTOR *variable*

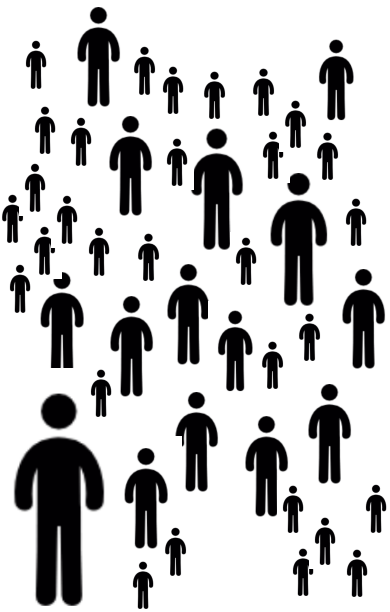
X ⟶ Age in Years

Random variable

RESPONSE *variable*

Y ⟶ dollars ($) In bank account

Population



Population parameters

$\beta_0, \beta_1, \sigma^2$

Hypothesis Test

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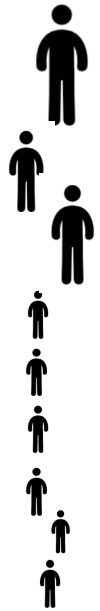

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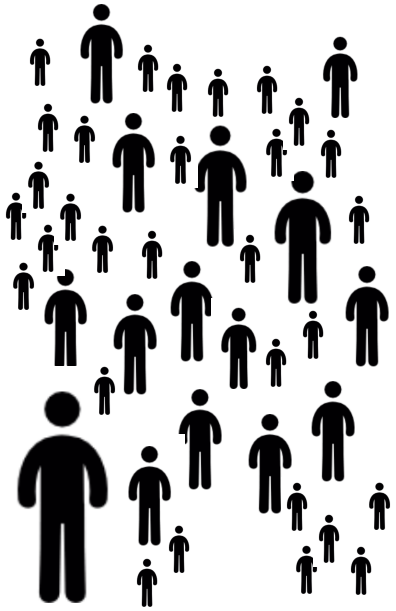
Age vs. Money

Population



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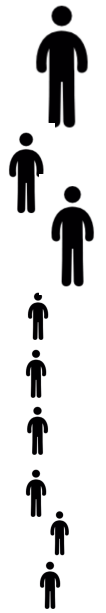

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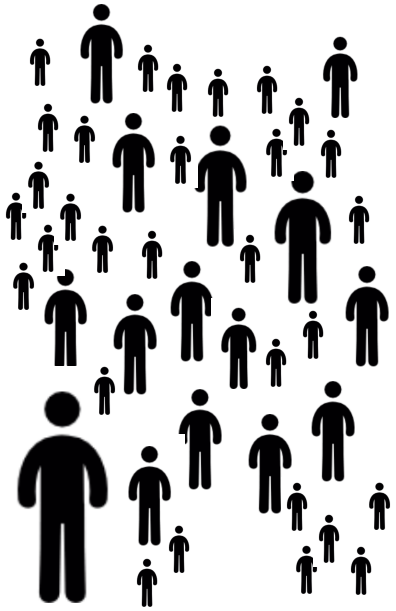
s = $\hat{\sigma}$ = residual SD = $\left\{(n-2)^{-1}\sum_{i=1}^{n} e_i^2\right\}^{1/2}$

```
> residuals <- y - b0_hat - b1_hat*x
> s <- sqrt( (1/(n-2))*sum(residuals^2) )
> s
[1] 15.5308
```

Age vs. Money

Population



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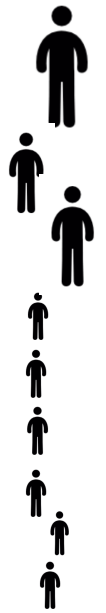

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95% C.I. for b_1 = $[b_1 - t_{n-2,0.975}\frac{\hat{\sigma}}{\sqrt{n-1}s_x}, \quad b_1 + t_{n-2,0.975}\frac{\hat{\sigma}}{\sqrt{n-1}s_x}]$

```
> b1_CI95 <- c(b1 - qt(0.975,n-2)*(s/(sqrt(n-1)*sx)) , b1 + qt(0.975,n-2)*(s/(sqrt(n-1)*sx)))
> b1_CI95
[1] 0.04946904 1.04676990
```

linear regression

Age vs. Money

fixed value

PREDICTOR *variable*

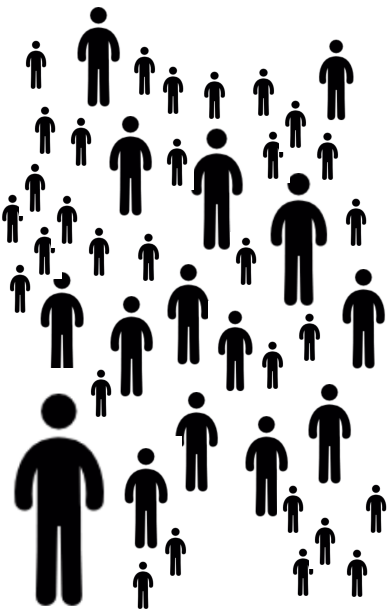
X ⟶ Age in Years

Random variable

RESPONSE *variable*

Y ⟶ dollars ($) In bank account

Population



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- Questions?

2.5.2 Derivations

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What is the $\text{subpopulation mean } \mu_Y(x) = \beta_0 + \beta_1 x$?

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The average amount of money (i.e. the expectation of random variable Y), among people aged “x” years old.

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Steps to get 95% C.I. for the subpopulation mean:

1. Consider the subpopulation mean, $\mu_Y(x)$, as a random variable.
2. Determine Var[$\mu_Y(x)$]
3. Define the standard error, $se(\hat{\mu}_Y(x))$, as an estimate of sqrt(Var($\mu_Y(x)$)) :
4. **95% C.I. = [** $\hat{\mu}_Y(x)$ **– c*** $se(\hat{\mu}_Y(x))$**,** $\hat{\mu}_Y(x)$ **+ c*** $se(\hat{\mu}_Y(x))$ **]**

2.5.2 Derivations

Steps to get 95% C.I. for the subpopulation mean:

1. Consider the subpopulation mean, $\mu_Y(x)$, as a random variable.

$\hat{\mu}_Y(x) = \hat{\beta}_0 + \hat{\beta}_1 x$ as a random variable is:

$$(2.52) \quad \hat{\mu}_Y(x) = (\overline{Y} - \hat{B}_1\overline{x}) + \hat{B}_1 x = \overline{Y} + \hat{B}_1(x - \overline{x}) = n^{-1}\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} a_i(x - \overline{x})Y_i = \sum_{i=1}^{n} c_i Y_i,$$

where

$$(2.53) \qquad c_i = n^{-1} + a_i(x - \overline{x}) = n^{-1} + \frac{(x - \overline{x})(x_i - \overline{x})}{(n-1)s_x^2}.$$

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2. Determine Var[$\mu_Y(x)$]

2.5.2 Derivations

Steps to get 95% C.I. for the subpopulation mean:

1. Consider the subpopulation mean, $\mu_Y(x)$, as a random variable.

2. **Determine Var[$\mu_Y(x)$]**

$$
\begin{aligned}
(2.63)\quad \mathrm{Var}\left[\hat{\mu}_Y(x)\right] &= \sum_{i=1}^{n} c_i^2 \mathrm{Var}\left(Y_i\right) = \sigma^2 \sum_{i=1}^{n} c_i^2 = \sigma^2 \sum_{i=1}^{n} \left\{ n^{-1} + (x-\overline{x})(x_i-\overline{x})/[(n-1)s_x^2] \right\}^2 \\
(2.64)\quad &= \sigma^2 \sum_{i=1}^{n} \left\{ n^{-2} + \frac{(x-\overline{x})^2 (x_i-\overline{x})^2}{[(n-1)s_x^2]^2} + 2n^{-1} \frac{(x-\overline{x})(x_i-\overline{x})}{[(n-1)s_x^2]} \right\} \\
(2.65)\quad &= \sigma^2 \left\{ n^{-1} + \frac{(x-\overline{x})^2 \sum_i (x_i-\overline{x})^2}{[(n-1)s_x^2]^2} + 0 \right\} \\
(2.66)\quad &= \sigma^2 \left\{ n^{-1} + \frac{(x-\overline{x})^2}{[(n-1)s_x^2]} \right\}.
\end{aligned}
$$

2.5.2 Derivations

Steps to get 95% C.I. for the subpopulation mean:

1. Consider the subpopulation mean, $\mu_Y(x)$, as a random variable.

2. Determine Var[$\mu_Y(x)$] $= \sigma^2\left\{n^{-1} + \frac{(x-\overline{x})^2}{[(n-1)s_x^2]}\right\}$.

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$$(2.64)\quad = \sigma^2 \sum_{i=1}^{n} \left\{n^{-2} + \frac{(x-\overline{x})^2(x_i-\overline{x})^2}{[(n-1)s_x^2]^2} + 2n^{-1}\frac{(x-\overline{x})(x_i-\overline{x})}{[(n-1)s_x^2]}\right\}$$

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3. Define the standard error $se(\hat{\mu}_Y(x))$ **, as an estimate of sqrt(Var(** $\mu_Y(x)$ **)) :**

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3. Define the standard error $se(\hat{\mu}_Y(x))$, as an estimate of sqrt(Var($\mu_Y(x)$)) :

$$se(\hat{\mu}_Y(x)) = \hat{\sigma} \times \sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{(n-1)s_x^2}}.$$

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4. **95% C.I. = [$\hat{\mu}_Y(x)$ – c* $se(\hat{\mu}_Y(x))$, $\hat{\mu}_Y(x)$ + c* $se(\hat{\mu}_Y(x))$]**

The 95% confidence interval for subpopulation mean $\mu_Y(x) = \beta_0 + \beta_1 x$ is

(2.43) $$\hat{\mu}_Y(x) \pm t_{n-2,0.975} \times se(\hat{\mu}_Y(x)), \quad \hat{\mu}_Y(x) = \hat{\beta}_0 + \hat{\beta}_1 x.$$

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$$se(\hat{\mu}_Y(x)) = \hat{\sigma} \times \sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{(n-1)s_x^2}}.$$

```
> myx <- 20
>  muhat_x <- b0+b1*myx
>  muhat_x
[1] 28.62758
> lowerCI <- muhat_x - qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> upperCI <-  muhat_x + qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> lowerCI
[1] 14.36777
> upperCI
[1] 42.88739
```

2.5.2 Derivations

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$$se(\hat{\mu}_Y(x)) = \hat{\sigma} \times \sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{(n-1)s_x^2}}.$$

```
> myx <- 40
>  muhat_x <- b0+b1*myx
>  muhat_x
[1] 39.58997
> lowerCI <- muhat_x - qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> upperCI <-  muhat_x + qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> lowerCI
[1] 27.06291
> upperCI
[1] 52.11703
```

2.5.2 Derivations

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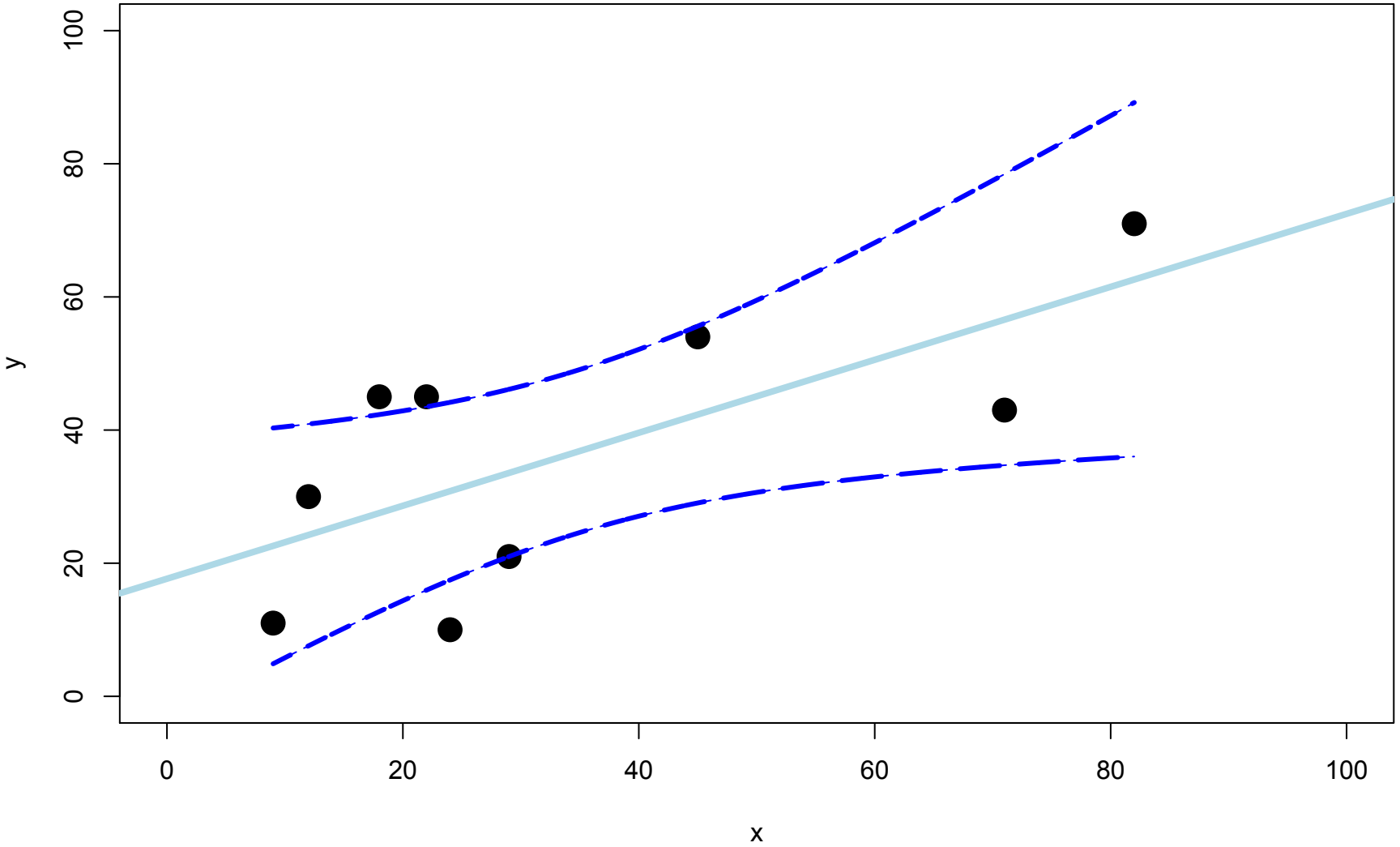
$$\hat{\mu}_Y(x) \pm t_{n-2,0.975} \times se(\hat{\mu}_Y(x)), \quad \hat{\mu}_Y(x) = \hat{\beta}_0 + \hat{\beta}_1 x. \tag{2.43}$$

$$se(\hat{\mu}_Y(x)) = \hat{\sigma} \times \sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{(n-1)s_x^2}}.$$

```
> myx <- 80
>  muhat_x <- b0+b1*myx
>  muhat_x
[1] 61.51475
> lowerCI <- muhat_x - qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> upperCI <-  muhat_x + qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> lowerCI
[1] 35.8075
> upperCI
[1] 87.22199
```

Age vs. Money



Solid Blue Line is the "subpopulation mean" for each value of x:
The average amount of money (i.e. the expectation of random variable Y), among people aged "x" years old.

2.5.2 Derivations

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```
> myx <- 100
>  muhat_x <- b0+b1*myx
>  muhat_x
[1] 72.47714
> lowerCI <- muhat_x - qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> upperCI <-  muhat_x + qt(0.975,n-2) * s * sqrt(1/n + ((myx-xbar)^2)/((n-1)*sx^2))
> lowerCI
[1] 37.67466
> upperCI
[1] 107.2796
```